

Asymptotic value of the minimal size of a graph with rainbow connection number 2^*

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Abstract

A path in an edge (vertex)-colored graph G , where adjacent edges (vertices) may have the same color, is called a rainbow path if no pair of edges (internal vertices) of the path are colored the same. The rainbow (vertex) connection number $rc(G)$ ($rvc(G)$) of G is the minimum integer i for which there exists an i -edge (vertex)-coloring of G such that every two distinct vertices of G are connected by a rainbow path. Denote by $\mathcal{G}_d(n)$ ($\mathcal{G}'_d(n)$) the set of all graphs of order n with rainbow (vertex) connection number d , and define $e_d(n) = \min\{e(G) \mid G \in \mathcal{G}_d(n)\}$ ($e'_d(n) = \min\{e(G) \mid G \in \mathcal{G}'_d(n)\}$), where $e(G)$ denotes the number of edges in G . In this paper, we investigate the bounds of $e_2(n)$ and get the exact asymptotic value. i.e., $\lim_{n \rightarrow \infty} \frac{e_2(n)}{n \log_2 n} = 1$. Meanwhile, we obtain $e'_d(n) = n - 1$ for $d \geq 2$, and the equality holds if and only if G is such a graph that deleting all leaves of G results in a tree of order d .

Keywords: edge(vertex)-coloring, rainbow path, rainbow (vertex) connection number, diameter, minimal graph

AMS subject classification 2010: 05C15, 05C35, 05C40

*Supported by NSFC No.11071130.

1 Introduction

A communication network consists of nodes and links which connect nodes. In order to prevent hackers, one can set a password in each link (node). To facilitate the management, one can require that the number of passwords is small enough such that any two nodes can exchange information by a sequence of links (nodes) which have different passwords. This problem can be modeled by a graph and studied by means of rainbow (vertex) connection.

All graphs in this paper are undirected, finite and simple. We refer to book [1] for graph theoretical notation and terminology not described here. A path in an edge (vertex)-colored graph G , where adjacent edges (vertices) may have the same color, is called a *rainbow path* if no pair of edges (internal vertices) are colored the same. An edge (vertex)-coloring of G with k colors is called *k-rainbow* if every two distinct vertices of G are connected by a rainbow path. The *rainbow (vertex) connection number* $rc(G)$ ($rvc(G)$) of G is the minimum integer i for which there exists an i -rainbow edge (vertex)-coloring of G such that any two distinct vertices of G are connected by a rainbow path. It is easy to see that $rc(G) \geq diam(G)$ and $rvc(G) \geq diam(G) - 1$ for any connected graph G , where $diam(G)$ is the diameter of G .

The rainbow connection number was introduced by Chartrand et al. in [5] which equivalents to the case that we set a password in each link. They considered the rainbow connection numbers of several graph classes (complete graphs, trees, cycles, wheels and complete bipartite graphs) and showed the following result.

Theorem 1. [5] (i) $rc(G) = 1$ if and only if G is a complete graph.

(ii) For integers s and t with $2 \leq s \leq t$,

$$rc(K_{s,t}) = \min\{\lceil \sqrt{s}t \rceil, 4\},$$

where $K_{s,t}$ is the complete bipartite graph with bipartition X and Y , such that $|X| = s$ and $|Y| = t$.

Krivelevich and Yuster in [7], and Schiermeyer in [10] investigated the relation between the rainbow connection number and the minimum degree

of a graph. Chandran et al. [7] studied the rainbow connection number of a graph by means of connected dominating sets. Basavaraju et al. in [2] evaluated the rainbow connection number of a graph by its radius and chordality (size of a largest induced cycle). In [3], Chakraborty et al. investigated the hardness and algorithms for the rainbow connection number, and got the following result.

Theorem 2. *Given a graph G , deciding if $rc(G) = 2$ is NP-Complete. In particular, computing $rc(G)$ is NP-Hard.*

It is well-known that almost all graphs have diameter 2. In [8], Li et al. showed that $rc(G) \leq 5$ if G is a bridgeless graph of diameter 2, and that $rc(G) \leq k + 2$ if G is a connected graph of diameter 2 with k bridges, where $k \geq 1$. For a detailed discussion regarding the origins of the problem, practical applications and a survey of results, see [9].

Let d and n be natural numbers, $d < n$. Denote by $\mathcal{G}_d(n)$ the set of all graphs of order n with rainbow connection number d . Define

$$e_d(n) = \min\{e(G) \mid G \in \mathcal{G}_d(n)\},$$

where $e(G)$ denotes the number of edges in G .

Because a network which satisfies our requirements and has as less links as possible can cut costs, reduce the construction period and simplify later maintenance, the study of this parameter is very interesting and significant. In this paper, we investigate the lower and upper bounds of $e_2(n)$ and get the exact asymptotic value for the minimal size of a graph with rainbow connection number 2. The following result is obtained:

Theorem 3.

$$\lim_{n \rightarrow \infty} \frac{e_2(n)}{n \log_2 n} = 1.$$

Krivelevich and Yuster in [7] introduced the concept of rainbow vertex connection number which is equivalent to the case that we set a password on each node. Let d and n be natural numbers, $d < n$. Denote by $\mathcal{G}'_d(n)$ the set of all graphs of order n with rainbow vertex connection number d . Define

$$e'_d(n) = \min\{e(G) \mid G \in \mathcal{G}'_d(n)\},$$

where $e(G)$ denotes the number of edges in G . The following result determines $e'_d(n)$.

Theorem 4. *Let d be an integer larger than 1. Then $e'_d(n) = n - 1$, and the equality holds if and only if G is such a graph that deleting all leaves of G results in a tree of order d .*

In the next section, we will prove Theorems 2 and 3.

2 The proofs of our main results

Since result on the minimal graphs with respect to the rainbow vertex connection number is easier to prove than that of the edge case, we first show Theorem 3.

Proof of Theorem 3 : Let G be such a tree that deleting all leaves of G results in a tree G' of order d . We now give G a d -rainbow vertex coloring as follows: color the vertices of G' by d distinct color and color all leaves of G by any used color. It is easy to check that this is a d -rainbow vertex coloring. Thus $e'_d(n) \leq n - 1$. On the other hand, any connected graph has at least $n - 1$ edges. Therefore $e'_d(n) = n - 1$.

Now, we consider the second part of this theorem. The necessity holds by the above argument. Conversely, let $G \in \mathcal{G}'_d(n)$ with $e(G) = n - 1$ and c be a d -rainbow vertex coloring. Suppose G has k leaves. Then we can give G an $(n - k)$ -rainbow vertex coloring by the above argument. Thus $rvc(G) = d \leq n - k$. On the other hand, we say $rvc(G) \geq n - k$. Let x and y be any pair of vertices that are not leaves. Since G is a tree, there exist two leaves, say x', y' , such that the unique path between x' and y' in G goes through x and y . Thus $c(x) \neq c(y)$. So $rvc(G) = d = n - k$, that is, $n - k = d$. Thus, by deleting all leaves from G , we get a tree G' with order d . \square

Now, we estimate the upper bound of $e_2(n)$ by constructing a family of graphs.

Lemma 1. *For $n \geq 2$*

$$e_2(n) \leq n \lceil \log_2 n \rceil - (\lfloor \log_2 n \rfloor - 1)^2.$$

Proof. For each integer $n \geq 2$, there exists an integer k such that $k + 2^{k-1} \leq n \leq k + 2^k$. Consider the number of edges in the complete bipartite graph $K_{k,n-k}$. We have

$$e(K_{k,n-k}) = k(n-k) \leq n \lceil \log_2 n \rceil - (\lfloor \log_2 n \rfloor - 1)^2.$$

Moreover, since $\lceil \sqrt[k]{n-k} \rceil \leq \lceil \sqrt[k]{2^k} \rceil = 2$, $rc(K_{k,n-k}) = 2$ follows from Theorem 1. Thus $e_2(n) \leq n \lceil \log_2 n \rceil - (\lfloor \log_2 n \rfloor - 1)^2$. \square

Consider a graph $G \in \mathcal{G}_2(n)$ with order n and maximum degree Δ . Pick a vertex $u \in V(G)$. Since $d(u) \leq \Delta$, there exist at most Δ vertices adjacent to u , and at most $\Delta(\Delta - 1)$ vertices at distance 2 from u . Since $\text{diam}(G) = 2$, we derive $n \leq 1 + \Delta + \Delta(\Delta - 1)$. Thus $\Delta \geq \sqrt{n-1}$. Since Δ is an integer, we get

$$\Delta \geq \lceil \sqrt{n-1} \rceil. \quad (1)$$

Next, we consider to get a lower bound for $e_2(n)$.

Lemma 2. (i)

$$e_2(n) \geq \min\left\{\frac{n}{2} \log_2 n, n \log_2 n - 4n\right\}.$$

(ii) If $n \geq 2^{17}$, then

$$e_2(n) \geq n \log_2 n - 2n.$$

Proof. Let G be a graph with diameter 2 and c be a 2-rainbow edge-coloring of G with colors blue and red. Set $k = \lceil (\log_2 \sqrt{n})^2 \rceil$ and denote by S the set of vertices with degrees less than k . Assume $S = \{u_1, u_2, \dots, u_s\}$, $T = V(G) \setminus S = \{u_{s+1}, u_{s+2}, \dots, u_{s+t}\}$, where $s+t = n$. By (1) and $k = \lceil (\log_2 \sqrt{n})^2 \rceil \leq \lceil \sqrt{n-1} \rceil \leq \Delta$, we know that T is nonempty. If $t = |T| \geq \frac{2n}{\log_2 \sqrt{n}}$, then

$$e(G) \geq \frac{1}{2} \sum_{v \in T} d_G(v) \geq \frac{1}{2} \frac{2n}{\log_2 \sqrt{n}} \lceil (\log_2 \sqrt{n})^2 \rceil \geq \frac{n}{2} \log_2 n,$$

we are done.

Suppose $t < \frac{2n}{\log_2 \sqrt{n}}$, that is, $s > n - \frac{2n}{\log_2 \sqrt{n}}$. Clearly, it is sufficient to show that $e(S, T) \geq n \log_2 n - 4n$.

For every u_i , $1 \leq i \leq s$, we define a vector as follows:

$$\alpha(u_i) = (b_{i,1}, b_{i,2}, \dots, b_{i,t}),$$

where

$$b_{i,j} = \begin{cases} 1 & \text{if } c(u_i u_{s+j}) \text{ is red;} \\ -1 & \text{if } c(u_i u_{s+j}) \text{ is blue;} \\ 0 & \text{if } u_i \text{ and } u_{s+j} \text{ is nonadjacent.} \end{cases}$$

Suppose $|N(u_i) \cap T| = a_i$, $1 \leq i \leq s$. Then $e(S, T) = \sum_{i=1}^s a_i$, where $e(S, T)$ denotes the number of edges between S and T . We now estimate the value of $e(S, T)$. For each $\alpha(u_i)$, we define a set B_i as follows: $B_i = \{\text{vectors obtained from } \alpha(u_i) \text{ by replace "0" of } \alpha(u_i) \text{ by "1" or "-1"}\}$. Because $|N(u_i) \cap T| = a_i$, we have $|B_i| = 2^{t-a_i}$ for each i , where $1 \leq i \leq s$. Set $B = \bigcup_{i=1}^s B_i$. Then B is a multiset of t -dimensional vectors with elements 1 and -1 . For each $\alpha \in B$, n_α denotes the number of α in B . We have the following claim.

Claim 1. *For each $\alpha \in B$, $n_\alpha \leq k^2 + 1$.*

Proof of Claim 1 : If Claim 1 is not true, that is, there exists a vector α , without loss of generality, assume $\alpha = (b_1, b_2, \dots, b_t)$, such that $n_\alpha \geq k^2 + 2$. Clearly, it is not possible that there exists some B_i such that B_i contains two α . Thus, there exist $k^2 + 2$ integers, without loss of generality, say $1, 2, \dots, k^2 + 2$, such that $\alpha \in B_r$, $1 \leq r \leq k^2 + 2$. we next show that for each i , $2 \leq i \leq k^2 + 2$, the distance between u_1 and u_i in $G[S]$ is at most 2. In fact, $c(u_1 u_{s+j}) = b_j = c(u_i u_{s+j})$ follows from the definition of B_1 and B_i . Thus there exists no rainbow path between u_1 and u_i through a vertex contained in T . So there must exist a rainbow path between u_1 and u_i with length at most 2 in $G[S]$. On the other hand, since $\Delta(G[S]) \leq k$, the number of vertices at distance 2 from u_1 is at most $k^2 + 1$, which is a contradiction. So, this claim is true.

By Claim 1, we know

$$\sum_{i=1}^s |B_i| \leq (k^2 + 1)2^t,$$

Since $|B_i| = 2^{t-a_i}$ for each i , $1 \leq i \leq s$,

$$\sum_{i=1}^s 2^{-a_i} \leq (k^2 + 1).$$

By the inequality between the geometrical and arithmetical means, we have

$$\sqrt[s]{2^{-e(S,T)}} = \sqrt[s]{2^{-\sum_{i=1}^s a_i}} \leq \frac{1}{s} \sum_{i=1}^s 2^{-a_i} \leq \frac{k^2 + 1}{s},$$

using the log function on both sides,

$$e(S, T) \geq s \log_2 s - s \log_2 (k^2 + 1).$$

Since $e(S, T)$ is monotonically decreasing in s and $s > n - \frac{2n}{\log_2 \sqrt{n}}$, we have

$$\begin{aligned} e(S, T) &\geq \left(n - \frac{2n}{\log_2 \sqrt{n}}\right) \log_2 \left(n - \frac{2n}{\log_2 \sqrt{n}}\right) \\ &\quad - \left(n - \frac{2n}{\log_2 \sqrt{n}}\right) \log_2 (\lceil (\log_2 \sqrt{n})^2 \rceil^2 + 1) \\ &= n \log_2 n - 4n. \end{aligned}$$

For (ii), take $k = \lceil (\log_2 n)^2 \rceil$. Since $n \geq 2^{17}$, we have $\lceil (\log_2 n)^2 \rceil \leq \lceil \sqrt{n-1} \rceil$. Thus T is nonempty by Ineq. (1). All the remaining arguments are similar to (i).

This completes the proof. □

Combining Lemmas 1 and 2, we know that Theorem 2 holds.

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